

PT-symmetric operators and metastable states of the 1D relativistic oscillators.

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Abstract

We consider the one-dimensional Dirac equation for the harmonic oscillator and the associated second order separated operators giving the resonances of the problem by complex dilation. The same operators have unique extensions as closed PT -symmetric operators defining infinite positive energy levels converging to the Schrödinger ones as $c \rightarrow \infty$. Such energy levels and their eigenfunctions give directly a definite choice of metastable states of the problem. Precise numerical computations shows that these levels coincide with the positions of the resonances up to the order of the width. Similar results are found for the Klein-Gordon oscillators, and in this case there is an infinite number of dynamics and the eigenvalues and eigenvectors of the PT -symmetric operators give metastable states for each dynamics.

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1 Introduction.

The harmonic oscillator is one of the fundamental dynamical systems both in classical and quantum theory [6] and the natural relativistic extensions have been investigated since a long time. In the relativistic quantum mechanics literature, therefore, one can find many different models that are considered relativistic oscillators [1–7]. These share the common property of reducing to the usual quantum oscillator in the Schrödinger limit but, otherwise, they present large differences in the way the interaction is constructed. Obviously different models have different dynamical and spectral properties: for instance scalar potentials have been added to vector potentials in a phenomenological way with the purpose of stabilizing the system and giving confinement, [8–11], or else the interaction has been obtained by adding to the free Dirac Hamiltonian a linear term in the position coordinate twisted by the Dirac β matrix [12–18]. The relevance of such models in physics is clear and goes beyond their use for an elementary description of the confinement mechanism, in view of the connection with QFT and of the application of the Dirac equations to new materials [19], where the relativistic effects are more easily measured.

We here call spinor and scalar relativistic oscillator, respectively, the Dirac and the Klein-Gordon equations with a quadratic electrostatic potential because these models are very realistic and, in principle, can be experimentally implemented. Although they do not satisfy the necessary

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conditions for the exactness of the Foldy-Wouthuysen transformations [20,21], it is obvious that the well known harmonic oscillator is reproduced in the Schrödinger limit. Despite their apparent simplicity, however, the study of the relativistic electrostatic oscillators is not so easy as it looks at first glance, but the results are satisfying.

Common wisdom tells us that strong instability is induced in quantum relativistic equations with unbounded potentials in the way described by the Klein paradox, [22]. Because of such instability the strongly bounded relativistic systems should present continuous spectrum and complex resonances. This picture is well signified by a result that Plesset established in 1932 [23] and that can be summarized, with a rather paradoxical formulation, as follows: the Dirac equation with a minimal coupling that involves an electrostatic “bounding” potential does not admit bound states. Of course, since bound states are present in the Schrödinger limit, the natural question that arises is what happens in the transition from a non-relativistic to a relativistic regime and in which sense this transition can be considered continuous in $1/c$. A first answer was given by Titchmarsh in [24] who studied in detail the perturbation treatment of the Dirac equation with piecewise linear vector potential. Due to the very cumbersome analytical calculations involved, the treatment was essentially restricted to the first perturbation order giving, however some more explicit informations on the properties of the continuous spectrum by investigating the Weyl function $m(\lambda)$ of the singular boundary value problem [25] in the complex plane. Very recently numerical investigations of the Dirac equation with a linear and a quadratic electrostatic potential were presented in [26,27], looking for a dissipative model: metastable states were found and the Schwinger pair production rate [28] was calculated in terms of the spectral concentration and in terms of the imaginary part of the resonances of the Dirac equation in external linear and quadratic potentials. In [26] the density of the states [29] was determined at finite values of $(1/c)$, finding a sum of Breit-Wigner lines whose width reproduced the pair production rate. In [27] the Dirac equation with a quadratic potential was revisited in the framework of the large order perturbation theory. The singular problem gave rise to asymptotic non oscillating series and the sum was calculated by the Distributional Borel (DB) method, [30–34], coupled with a Padé approximation for the DB transform, finding complete agreement with [26]. Moreover the spectral problem was clearly formulated stressing the reasons for the dissipative choice. It was also shown that this choice leads to the study of non self-adjoint operators defined by complex dilation: the imaginary part of the complex energy levels, interpreted as resonances, again reproduces the pair production.

Although exact bound states do not exist, in this paper we show that particular metastable states can be identified. They are uniquely obtained from eigenvalues of the decoupled equations of Titchmarsh. At least for a small ratio of the interaction to the mass energy they are well approximated. The role of the corresponding levels is similar to the role of the real part of the resonances. In this case we have both and it is our purpose to compare them. Actually, for the metastable states we have more exact results. In fact the Klein-Gordon like Hamiltonians, obtained by Titchmarsh in [24] (see also [35]) from the separation of the Dirac system of equations – and hereby referred as to Titchmarsh Hamiltonians – are uniquely defined closed operators with a discrete spectrum that gives the metastable levels of the Dirac equation. The Titchmarsh Hamiltonians exhibit what in current literature is known as the PT -symmetry, where P is the parity transformation and T the complex conjugation [1,36,37]: our paper, therefore, adds also a meaningful contribution to the discussion on the physical meaning of the PT -symmetric operators that still keep arising great interest (see, for instance, [38–42]). The metastable levels are studied in the next two sections. In section 2 we use the methods of the functional analysis supported by some recent results coming from the study of anharmonic quantum oscillators [36,43] and we

give formal statements of our analytical results. We also show the possible role of the metastable levels as stationary levels of the separated PT -symmetric dynamics. In section 3 we then present the complex extensions of the position variables we have used, that have been introduced for two reasons: a better variational approximation and the connection with known results, mainly those of references [36, 43]. In the final section we present a discussion of the numerical results connecting the metastable levels to the resonances of the model. We have thus calculated some perfectly defined energy levels of the Titchmarsh Hamiltonians stable at the Schrödinger limit and we have established their relationship to the real part of the resonances determined in [27] by the the DB sum: we have thus proved that the difference is of the second order in the imaginary part of the resonances themselves and it can therefore be ascribed to the pair production rate, as it had to be expected on a physical ground. We have found that the best numerical way, reaching the very high precision necessary in order to compare the asymptotic behavior in a parameter $\Omega \sim O(1/c^2) \rightarrow 0$ with the DB sum of the power series expansion [27] is provided by a specialization of the Rayleigh-Ritz scheme [44] obtained through the matrix moments method [45, 46]. This method could also be used for higher values of Ω where, however, other approaches could be equally or even more efficient [26, 47]

2 The Dirac and the Klein-Gordon one-dimensional oscillators.

Let us consider the one-dimensional Dirac equation in an electrostatic potential $V(x)$ [24, 35]. Using the notations of ref. [24] we assume a two-component spinor wave function of the form $X = {}^t[X_2(x), X_1(x)]$ so that the explicit form of the Dirac equation reads

$$\begin{aligned} \frac{1}{c}(W + mc^2 - V(x))X_1(x) - \hbar \frac{d}{dx} X_2(x) &= 0 \\ \hbar \frac{d}{dx} X_1(x) + \frac{1}{c}(W - mc^2 - V(x))X_2(x) &= 0 \end{aligned} \quad (2.1)$$

We therefore see that the ‘large’ component of the spinor is $X_2(x)$. In the following we will assume

$$V(x) = \frac{1}{2} m\omega^2 x^2.$$

We rescale the spatial coordinate as

$$x \mapsto \left(\frac{m\omega}{\hbar}\right)^{1/2} x$$

so to make it dimensionless and we define the equally dimensionless parameters

$$\Omega = \left(\frac{\hbar\omega}{4mc^2}\right), \quad E = \frac{2}{\hbar\omega} (W - mc^2).$$

From (2.1) we then get the system

$$\begin{aligned} \frac{d}{dx} X_2(x) - \sqrt{\Omega} \left(E - x^2 + \frac{1}{\Omega}\right) X_1(x) &= 0 \\ \frac{d}{dx} X_1(x) + \sqrt{\Omega} (E - x^2) X_2(x) &= 0 \end{aligned} \quad (2.2)$$

In order to reduce the system (2.2) to separate second order equations, it is convenient to introduce the linear combinations

$$\psi_+(x) = \frac{1}{\sqrt{2}} (X_1(x) + iX_2(x)) \quad \psi_-(x) = -\frac{i}{\sqrt{2}} (X_1(x) - iX_2(x)), \quad (2.3)$$

yielding

$$\begin{aligned} \frac{d}{dx} \psi_+(x) - i\sqrt{\Omega} (E - x^2) \psi_+(x) - \frac{i}{2\sqrt{\Omega}} (\psi_+(x) + i\psi_-(x)) &= 0 \\ \frac{d}{dx} \psi_-(x) + i\sqrt{\Omega} (E - x^2) \psi_-(x) + \frac{1}{2\sqrt{\Omega}} (\psi_+(x) + i\psi_-(x)) &= 0. \end{aligned} \quad (2.4)$$

Solving the first equation (2.4) in $\psi_-(x)$,

$$\psi_-(x) = -2\sqrt{\Omega} \frac{d}{dx} \psi_+(x) + i(2\Omega(E - x^2) + 1) \psi_+(x)$$

and substituting in the second equation, we finally find for $\psi_+(x)$ the second order equation

$$\tilde{H}_+(\Omega, E) \psi_+(x) = \lambda \psi_+(x)$$

where $\lambda = \lambda(\Omega, E) = E + \Omega E^2$ and where the Titchmarsh Hamiltonian

$$\tilde{H}_+(\Omega, E) = -\frac{d^2}{dx^2} - 2i\sqrt{\Omega}x + (1 + 2E\Omega)x^2 - \Omega x^4 \quad (2.5)$$

is a formal operator in the Hilbert space of the states $\mathcal{H} = L^2(\mathbb{R})$. Here and in the following we will always take $\sqrt{\Omega} > 0$.

The assumption of a real energy E will prove to be self-consistent with the boundary value problem that turns out to be complete, in the sense that the formal operator $\tilde{H}_+(\Omega, E)$ extends uniquely to a closed Hamiltonian $H_+ = H_+(\Omega, E)$ (with discrete spectrum). The unstable term $-\Omega x^4$ is still present in H_+ , suggesting that the Klein paradox arguments could be brought to bear: the imaginary term $-2i\sqrt{\Omega}x$, however, makes the problem complete and the PT -symmetric extension of H_+ unique. Moreover the unstable quartic term can be easily dealt with. Indeed in [36] a direct demonstration was given of the equality of the eigenvalues of a pair of quantum systems: the double well and the unstable anharmonic oscillator defined by complex translation, both of them, obviously, making sense and being well defined. In [43], then, a general proof was presented that the spectrum of H_+ is discrete and positive:

$$\sigma(H_+) = \{\lambda_n > 0\}_{n \in \mathbb{N}}.$$

The equivalent adjoint Hamiltonian is

$$H_- = PH_+P = TH_+T = H_+^*, \quad (2.6)$$

where P is the parity transform and T is the time reversal.

Let us also notice that the non self-adjointness of the Hamiltonians has a physical meaning: it allows to define uniquely and to distinguish the H_{\pm} operators. Finally, the labeling of the levels is the same we have at the Schrödinger limit, in the hypothesis that only one level is associated to one eigenvalue (see equation (2.8) here below). We shall argue that this is the case and, moreover,

we expect and we prove the existence of infinite positive levels, stable at $\Omega = 0$. Let us now express our results in a formal way.

THEOREM 2.1 *Let $\Omega > 0$, $E \in \mathbb{R}$, then it is uniquely defined the closed PT -symmetric extension with discrete spectrum, $H_+(\Omega, E)$ of the operator $\tilde{H}_+(\Omega, E)$. The closed operator $H_-(\Omega, E)$ is then defined from (2.6).*

PROOF. We uniquely define the operator H_+ by the L^2 conditions at $\pm\infty$. The asymptotic behavior of the fundamental solution for $x \rightarrow +\infty$ is

$$\Psi_+(x) \sim \frac{1}{x} \exp(iS(x) - \ln(x)) = \frac{1}{x^2} \exp(iS(x)),$$

where

$$S(x) = \int^x \sqrt{\Omega y^4 - (1 + 2E\Omega)y^2} dy \sim \sqrt{\Omega} \frac{x^3}{3} - \frac{(1 + 2E\Omega)}{2\sqrt{\Omega}} x, \quad (2.7)$$

the other having the behavior

$$\Psi_-(x) \sim \frac{1}{x^2} \exp(iS(x)) \quad \text{as } x \rightarrow -\infty. \quad \square$$

THEOREM 2.2 *Let $E \in \mathbb{R}$, $\Omega > 0$, and fix the index state \pm , there are infinite positive simple eigenvalues $\lambda_n(\Omega, E)$ $n \in \mathbb{N}$ of the Titchmarsh operators for the harmonic oscillator. For each $\Omega > 0$, there are infinite positive energy levels $E_n(\Omega)$ of the spectrum of the Titchmarsh operators.*

PROOF. Let us fix $\Omega > 0$, $n \in \mathbb{N}$ and recall that, according to [43], for any $E \in \mathbb{R}$ we have positive eigenvalues $\lambda_n(E)$ of $H_\pm(E)$. We thus signify by “level” any generalized eigenvalue $E_n > 0$, solution of the implicit equation $\Omega E^2 + E = \lambda_n(E)$, namely

$$E = \frac{1}{2\Omega} (\sqrt{1 + 4\lambda_n(E)\Omega} - 1) \quad (2.8)$$

The existence of a solution of (2.8) is proved in the following way. By real rescaling, and using the perturbation theory, we obtain, for fixed n , $\Omega > 0$ and large positive E ,

$$\lambda_n(E) = \sqrt{1 + 2E\Omega} (2n + 1 + O(\Omega/(1 + 2E\Omega)^{3/2})) \ll E + \Omega E^2. \quad (2.9)$$

The same behavior applies for $E > 0$ and small Ω , so that in this regime the solution of (2.8) is unique. \square

Let us now consider how these results can be brought to bear to the study of the Dirac equation. From (2.3) we see that the components $X_1(x)$ and $X_2(x)$ are linear combinations of two solutions $\psi_\pm(x)$ of the second order equations. The eigenstates of the two Titchmarsh operators are independent because they are complex conjugate with non-vanishing real and imaginary parts, as it can be seen from the equations and their asymptotic behaviors. If we take a pair of eigenfunctions $(\psi_+(x), \psi_-(x))$ of the separated equations such that $\psi_+(x) = i\overline{\psi_-(x)}$, this pair gives directly metastable states of the full problem, whereas the resonances give metastable states only by a cut-off: it is relevant that we have more exact informations [36, 43] on the metastable states than on the resonances. In the non-relativistic limit $\Omega \rightarrow 0$, the energy levels $E_n(\Omega)$ tend to the eigenvalues E_n of the Schrödinger Hamiltonian and both the states tend to the real Schrödinger eigenstate ψ_n corresponding to E_n . Recalling again (2.3) we also remark that, for the metastable states, $X_2 \sim \text{Re } \psi_+$ and $X_1 \sim \text{Im } \psi_+$. The sizes of the large and small components of the spinors, therefore, turn out to be different by many orders of magnitude, providing an approximation to

levels of the metastable states which is very natural and more accurate than the one we could obtain starting from the Schrödinger eigenvalues and perturbing them by means of the relativistic terms: indeed we the metastable levels are approximated by the DBS that gives the sum of the complete perturbation series in $(1/c^2)$.

Take then the real operator

$$\tilde{K}(\Omega, E) = \tilde{H}(\Omega, E) + 2i\sqrt{\Omega}x. \quad (2.10)$$

This is just the Klein-Gordon Hamiltonian with quadratic electrostatic potential and it is defined as a closed PT -symmetric operator $K_+(\Omega, E)$ by the behavior of the fundamental solutions,

$$\Phi_{\pm}(x) \sim \frac{1}{|x|} \exp(iS(x)) \quad (2.11)$$

as $x \rightarrow \pm\infty$. K_+ has positive eigenvalues λ_n for any real parameter E [36, 43]. Again the equivalent adjoint Hamiltonian,

$$K_- = PK_+P = TK_+T = K_+^*$$

is obtained by parity transform. Following the same proof of Theorem 2, we can show that there are *infinite positive energy levels* as in the Dirac case. The boundary conditions are similar to the resonance ones, but there are always *mixed conditions* Gamow, anti Gamow at $\pm\infty$. We define as the PT -symmetric pair of dynamics the dynamics generated by the two Hamiltonians K_{\pm} . It is interesting to note that the energy levels are the same for the two dynamics, and that the two states are complex conjugate.

REMARK 2.1 Let us stress that the small difference between \tilde{H}_+ and \tilde{K}_+ is, however, relevant: the formal operator \tilde{K} *is in fact not uniquely implemented as a closed operator*. In particular, we have infinite self adjoint extensions with discrete spectrum. Moreover, there is the closed extension defined by the Gamow condition at $\pm\infty$, with eigenvalues considered resonances, in a generalized sense, given by the Distributional Borel sum of the perturbation series.

In perfect analogy with the H_{\pm} Hamiltonians, we consider a pair of isospectral PT -symmetric Hamiltonians K_{\pm} with positive simple eigenvalues. In this problem, where *the* physical dynamics does not exists because is not unique, it is appropriate to consider as physical levels, in first approximation, the values taken from the common eigenvalues of the pair of Hamiltonians K_{\pm} . The corresponding eigenstates of the pair of Hamiltonians K_{\pm} and their combinations, as the real means, are metastable states for all the infinite dynamics of the problem. The real part of the resonances looks less physical of the eigenvalues of K_{\pm} , even if are very similar for small Ω . Let us remark that probably it is not appropriate to impose the same principles of QM to the relativistic QM, where the pair production effect exists. But it is appropriate to pay the minimum price.

For real energy E , we have computed the eigenvalues of (2.5) by a variational method called the matrix moment method. Since each eigenvalue λ_n is positive, [43], we get the corresponding positive energy level (2.8) as in the Dirac case.

3 Definition of the Hamiltonians by subdominant behavior on two disjoint sectors.

In this section we will give the prescription for a non-ambiguous definition of the possible operators we can obtain from the formal operator $\tilde{H}(\Omega, E)$ with positive parameters $(\sqrt{\Omega}, E)$. We follow the

methods and the terminology of references [36, 48]. Let us fix $\arg(ix) = \phi$, and consider the six sectors:

$$S_j = \{-\pi/6 < (\phi - j\pi/3) < \pi/6\}, \quad -2 \leq j \leq 3.$$

In particular, we define subdominant in the sector S_j , $-2 \leq j \leq 3$, the solution Ψ_j of the second order equation, $\tilde{H}(\Omega, E)\Psi_j = 0$, with the principal behavior for $\phi = j\pi/3$,

$$\ln(\Psi_j(x)) = -\sqrt{\Omega} \frac{|x|^3}{3} (1 + O(|x|^{-2})) \text{ as } |x| \rightarrow \infty. \quad (3.1)$$

In certain cases, the L^2 behavior extends partially to the closure \bar{S}_j of the angular sectors. In particular, the PT -symmetric Hamiltonian $H_+(\Omega, E)$ is defined by the subdominant behavior on the pair of sectors (S_{-2}, S_2) . Actually, the solutions respectively subdominant on the two sectors can be identified with the two fundamental solutions at $\pm\infty$.

Let us now examine the complex contours that can be taken according to the operators we want to define.

(i) *Complex translation and distortion.* Let us start again from $H = \tilde{H}_+(\Omega, E)$ for positive $(\sqrt{\Omega}, E)$, as above. Consider next the complex translation

$$\psi(x) \mapsto T_z \psi(x) = \psi(x + z),$$

and the translated operator

$$H^z = T_z H T_{-z},$$

$$H^z(\Omega, E) = -\frac{d^2}{dx^2} - \Omega E^2 - E - 2i\sqrt{\Omega}(x + z) + (1 + 2E\Omega)(x + z)^2 - \Omega(x + z)^4. \quad (3.2)$$

For $z = iy$, $y > 0$, H^z is uniquely defined as a closed operator by the fundamental solutions

$$\Psi_{\pm}^z(x) \sim \frac{1}{|x|^2} \exp(iS(x + z)), \quad (3.3)$$

as $x \rightarrow \pm\infty$, respectively. A relevant observation is in order. As the fundamental solutions are coincident with the subdominant solutions on the pair of sectors (S_{-2}, S_2) , we can prove that all the translated operators are isospectral with the operator defined by the subdominant condition on the two sectors above. Indeed it is easy to see that the fundamental solutions of the translated operators are the translated of the fundamental solutions of $H_+(\beta, E)$, with the behavior

$$\Psi_{\pm}(x) \sim \frac{1}{|x|^2} \exp(iS(x)), \quad (3.4)$$

as $x \rightarrow \pm\infty$, respectively. Since the eigenvalues are the zeros of the Wronskian of the two fundamental solutions, we can conclude that the translated operators are isospectral with the operator $H_+(\Omega, E)$ itself. In the numerical calculations the translation parameter y will be used as a variational parameter in the application of the Rayleigh-Ritz method. We could also consider the PT -symmetry conserving distortion of entire functions,

$$\psi(x) \mapsto D\psi(x) = \sqrt{z'(x)} \psi(z(x)),$$

where $z(x) = x(1 + i \tan(\pi/6) x \sqrt{1 + x^2})$, [49], yielding strictly sectorial operators,

$$H_+^D(\Omega, E) = D H_+(\Omega, E) D^{-1}.$$

These will be treated elsewhere.

(ii) *Complex dilation.* Let $H = \tilde{H}_+(\Omega, E)$ as above, with $\Omega > 0$. Consider first the complex dilation

$$\psi(x) \mapsto T_\theta \psi(x) = \psi(x \exp(i\theta)) = \psi^\theta(x),$$

and the operator

$$\tilde{H}_\theta(\Omega, E) = T_\theta \tilde{H}(\Omega, E) T_{-\theta}, \quad (3.5)$$

where $\psi(x)$ is restricted to the dense set of L^2 functions analytic on an angular sector

$$\left| \frac{\operatorname{Im} z}{\operatorname{Re} z} \right| < \tan(\theta_0) \quad \text{with} \quad 0 < |\theta| < \theta_0 < \pi/6.$$

Such operators are defined by the subdominant behavior on the two pairs of sectors (S_{-2}, S_1) and (S_{-1}, S_2) and they are isospectral to the dilated ones with negative and positive θ respectively. For positive θ the fundamental solutions at $\pm\infty$ have the behavior

$$\ln(\Psi_\pm^\theta(x)) = i S(x \exp(i\theta)) + O(\ln(|x|)), \quad (3.6)$$

as $x \rightarrow \pm\infty$, respectively. For negative θ the fundamental solutions at $\pm\infty$ are

$$\ln(\Psi_\pm^\theta(x)) = -i S(x \exp(i\theta)) + O(\ln(|x|)). \quad (3.7)$$

The two DB complex conjugate sums [30–34] apply to the energy levels of these operators defined by the subdominant behaviors on the two pairs of sectors (S_{-2}, S_1) and (S_{-1}, S_2) , respectively. The application of the perturbation theory leading to the DB sums has been thoroughly studied in [27] by numerical methods, specializing, in particular, to the calculation of the pair production rate obtained from the imaginary part of the resonance.

4 Results and conclusions.

In order to compare the spectra of different boundary value problems, instead of integrating the differential equations as in [26], we have calculated the eigenvalues using a slight modification of the classical Weinstein-Aronszajn method [44] obtained from the theory of matrix moments, [45], joint to the Newton procedure for finding the zeros of a function. In fact, due to the Jacobi form (4.1) of the Hamiltonian matrix in the base of the occupation number states, this is equivalent to a Rayleigh-Ritz variational scheme made more efficient through the matrix moments that shorten the computational time and increase the precision of the calculation.

We report, in Figure 1 the lowest energy levels of the Dirac electrostatic oscillator for different values of the parameter Ω .

We consider the translated, real and dilated operators uniquely defined on $L^2(\mathbb{R})$. These are compressed on the space \mathcal{H}_n spanned by the first n Hermite eigenfunctions $\{\psi_j\}_{j \leq n}$ of the Schrödinger Hamiltonian $H_0 = p^2 + \sigma^2 x^2$, where σ is a variational parameter. The matrix elements H_{ik} of the fourth degree polynomial in x are obviously dealt with by using the recurrence relation

$$x h_k = (1/2) h_{k+1} + k h_{k-1}$$

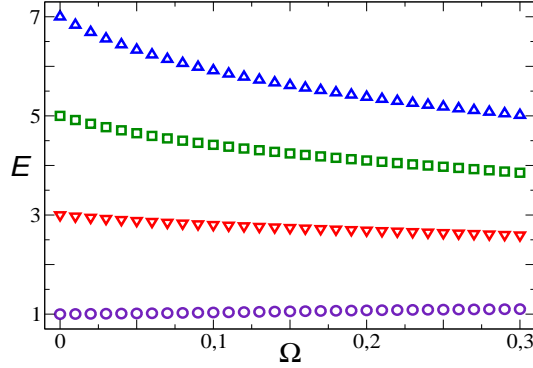


Figure 1: The lowest energy levels of the Dirac oscillators for different values of Ω . Circles refer to the lowest positive energy level. Down triangles, squares and up triangles refer to the first, second and third excited levels respectively.

for the Hermite polynomials $h_r = h_r(x)$. The explicit relevant relations we are using read

$$\begin{aligned}
x^2 h_k &= (1/4) h_{k+2} + (k+1/2) h_k + n(k-1) h_{k-2} \\
x^3 h_k &= (1/8) h_{k+3} + (3/4)(k+1) h_{k+1} + (3/2) k^2 h_{k-1} + k(k-1)(k-2) h_{k-3} \\
x^4 h_k &= (1/16) h_{k+4} + (1/4)(3+2k) h_{k+2} + (3/4)(2k^2+2k+1) h_k + k(2k-1)(k-1) h_{k-2} \\
&\quad + k(k-1)(k-2)(k-3) h_{k-4}
\end{aligned}$$

Thus the general form of the matrix reads

$$(H_{ik}) = \begin{pmatrix} A_0 & B_0 & 0 & 0 & \dots \\ C_0 & A_1 & B_1 & 0 & \dots \\ 0 & C_1 & A_2 & B_2 & 0 \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (4.1)$$

where the symbols A_n , B_n and C_n represent 4×4 blocks given by

$$\begin{aligned}
(A_n)_{ik} &= H_{4n+i, 4n+k} \\
(B_n)_{ik} &= H_{4n+i, 4(n+1)+k} \\
(C_n)_{ik} &= H_{4(n+1)+i, 4n+k} = H_{4n+k, 4(n+1)+i} = (B_n)_{ki}
\end{aligned}$$

with $n = 0, 1, 2, \dots$ and $i, k = 0, 1, 2, 3$.

The first $4n$ eigenvalues can be approximated by the eigenvalues of the Hamiltonian matrix truncated at the $4n$ -th order. From the matrix moment theory [45] these eigenvalues are given by the zeroes of the determinant of the matrix polynomials $P_n = P_n(\lambda)$, recursively defined in the following way:

$$\begin{aligned}
P_0 &= I_4 \\
P_1 &= B_0^{-1}(\lambda I_4 - A_0) \\
P_n &= B_{n-1}^{-1}(\lambda P_{n-1} - C_{n-2} P_{n-2} - A_{n-1} P_{n-1}), \quad (4.2)
\end{aligned}$$

I_4 being the 4×4 identity matrix. The solutions of $\det P_n(\lambda) = 0$ are indeed the $4n$ -th order Rayleigh-Ritz approximants that give upper bounds for the eigenvalues. Although lower bounds

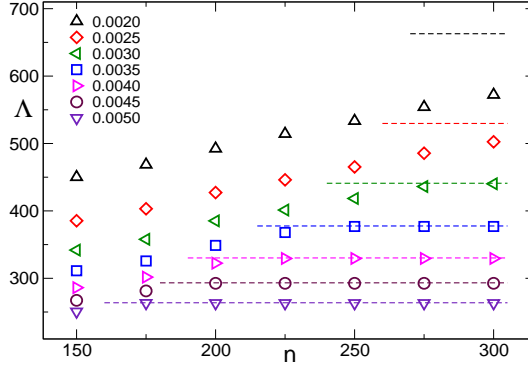


Figure 2: The plot of $\Lambda = -\ln(E_{t0} - \text{Re}(E_{d0}))$ vs. the number n of iterations of (4.2). Different symbols correspond to different values of Ω . The horizontal dashed lines give $2\text{Im}(E_{d0})$ for the corresponding Ω .

for the eigenvalues do not prove to be useful in our present context, it could however be observed that they can be obtained by this same method, looking for the zeroes of the determinant of the matrix polynomial [46]

$$P_n(\lambda) - P_n(0)P_{n-1}^{-1}(0)P_{n-1}(\lambda)$$

Let us now present the numerical results. We denote by “ r , t , d ” the quantities respectively related to the spectrum of the real, translated and dilated operators (2.10), (3.2) and (3.5) previously discussed. In Table 1, for different values of Ω , we give the complex d -eigenvalue E_{d0} with lowest positive real part, reducing to the ground level in the non-relativistic limit. Since the effect

Ω	E_{d0}
0.0020	$1.0005017620 + i \ 1.17374083059 \text{e-}144$
0.0025	$1.0006277579 + i \ 9.42079110945 \text{e-}116$
0.0030	$1.0007539782 + i \ 1.72376665081 \text{e-}96$
0.0035	$1.0008804241 + i \ 9.77543924661 \text{e-}83$
0.0040	$1.0010070969 + i \ 2.00211928567 \text{e-}72$
0.0045	$1.0011339978 + i \ 2.08165603853 \text{e-}64$
0.0050	$1.0012611278 + i \ 5.36447802132 \text{e-}58$

Table 1: The lowest d -eigenvalue E_{d0} for varying Ω .

to be highlighted is really tiny, these eigenvalues have been calculated with great accuracy, both in the arithmetic precision and in the number of iteration of the recurrence relation (4.2). The largest value of the latter has been taken to be 300: although a large number of decimal figures is already stabilized by few iterations, a very high precision is however necessary for a comparison of the t with the d eigenvalues, that we show here below. As expected, the results obtained in [26, 27] when studying the resonances of the Dirac equation by the spectral concentration and by the DB sum, as well as the asymptotic behavior of the imaginary part, are confirmed by the much more precise data given in Table 1. In the application of the Rayleigh-Ritz method we have used as variational parameters the size of the imaginary translation y introduced in item (3.3) and the ‘frequency’ σ of the operator H_0 . There has been numerical evidence that the optimal values of

these parameters are $1 \lesssim y \lesssim 5$ and $1 \lesssim \sigma \lesssim 2$. In Figure 2 we plot

$$\Lambda = -\ln(E_{t0} - \text{Re}(E_{d0}))$$

for a different number n of iterations of (4.2) in order to test the stabilization of the data with n . We see that the saturation for decreasing values of Ω requires increasing values of n : it appears, however, that $n = 300$ is already sufficient for $\Omega = 0.0030$ and higher. The final value is almost coincident with minus twice the logarithm of the imaginary part of the eigenvalue E_d , represented by the dashed horizontal lines.

REMARK 4.1 This means that each energy level is given by the distributional Borel sum of the perturbation series modulo a correction of the second order on the pair production effect (see [27]).

This property has been numerically checked by calculating the ratio $\Lambda/(-2 \ln \text{Im}(E_{d0}))$ for $0.0030 \leq \Omega \leq 0.0050$. A simple quartic Lagrangian interpolation on the data thus obtained gives 0.999885 as a limiting value of the ratio.

We finally want to consider the influence of the imaginary term proportional to $\sqrt{\Omega}$ in (3.2). We have therefore calculated the difference of the first t and r eigenvalues and the ratio

$$\kappa = (E_{t0}(\Omega) - E_{r0}(\Omega))/\Omega.$$

A least square computation on the data for $0.002 < \Omega < 0.005$ gives $\kappa = 0.9999539755 + 0.0284823904\Omega$, providing a numerical evidence that the difference of the previous eigenvalues vanishes in the limit of a vanishing Ω .

We have compared the differences of the first excited minus the fundamental level for the t and r operators:

$$\delta = [(E_{t1}(\Omega) - E_{t0}(\Omega)) - (E_{r1}(\Omega) - E_{r0}(\Omega))]/\Omega.$$

finding a more than linear vanishing behavior with vanishing Ω .

To conclude, in this paper we have completed the rigorous analysis of the basic model of the harmonic oscillator in 1D Dirac and Klein-Gordon theory, using recent mathematical results concerning the anharmonic oscillators and considering the relevance of the physical symmetries [36,37]. As we said, the result is satisfying, since it brings it one step closer to the QFT symmetries. We have also shown that discrete energy values, the exceptional points, can be obtained by the approximation of decoupling two equations, but without the presence of a scalar potential, whose physical interpretation is usually uncertain. The philosophy could be that: in relativistic QM it is not possible or convenient to keep all the principles of QM, but we can try to minimize the effects of the pair production. We have PT-symmetric energy levels comparable to the non relativistic ones. This way, the Klein Gordon equation appears more similar to the Dirac one. In this case, the PT-symmetric eigenvalues appears more physical than the infinite ones of the infinite selfadjoint Hamiltonians. The levels of the 2D and 3D Dirac oscillators will appear in a following paper.

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